Lecture 34

16.3 - The Fundamental Theorem for Line Integrals

This section begins the theme of the rest of the course: What is the fundamental theorem of calculus in higher dimensions?

FTOC: $\int_a^b F'(x)dx = F(b) - F(a)$

Let's consider a gradient vector field ∇f where f is a function of 2 or 3 variables.

Fundamental Theorem of Line Integrals

Let C be a smooth curve parametrized by $\vec{r}(t)$, $a \le t \le b$, and f a C^1 function (∇f is continuous). Then

$$\boxed{\int_a^b \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

If C is a curve starting at A and ending at B, sometimes we write:

$$\int_C \nabla f \cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A)$$

proof

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$\overset{\text{(Chain Rule)}}{=} \int_a^b \frac{d}{dt}\left[f(\vec{r}(t))\right] dt \overset{\text{(Regular FTOC)}}{=} f(\vec{r}(b)) - f(\vec{r}(a))$$

$\square$

So, this theorem can be used to make computations easier in some cases.

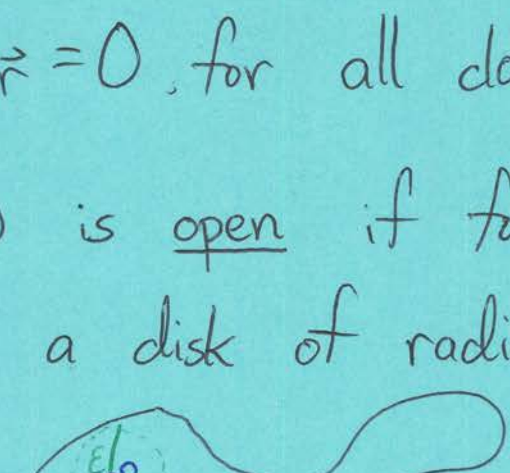
First, something else. We know from Friday that $\int_C y\,dx - x\,dy + z\,dz$ depends on the path, but the theorem seems to imply that the integral does not depend on the path.

If $\vec{F}$ is continuous on D and C_1 and C_2 are paths in D, we say the line integral is <u>independent of path</u> if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1 & C_2 with the same end points. So, the theorem says that <u>line integrals of conservative vector fields are independent of path</u>.

<u>Def</u>: A curve is called <u>closed</u> if it starts and ends at the same point. ($\vec{r}(a) = \vec{r}(b)$, if $\vec{r}(t)$, $a \le t \le b$ is a parametrization of the curve)

<u>Thm</u>: $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for all closed paths C.

<u>Def</u>: A set D is <u>open</u> if for every point P in D, we can fit a disk of radius ε with center P inside D:



<u>Def</u>: A set D is <u>connected</u> if any two points in D can be connected by a path in D.

Finally,

<u>Thm</u>: Suppose $\vec{F}$ is a vector field which is continuous on an open connected set D. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is conservative.

So, we have all this great stuff about conservative vector fields, but how do we check if a vector field is conservative?

For a conservative vector field $\vec{F} = \langle P, Q \rangle$, we have $\vec{F} = \nabla f = \langle f_x, f_y \rangle$. Remember, if f is C^2, then $f_{xy} = f_{yx}$, so if $\vec{F} = \langle P, Q \rangle$ is conservative

$P = f_x$ & $Q = f_y \Rightarrow P_y = Q_x$.

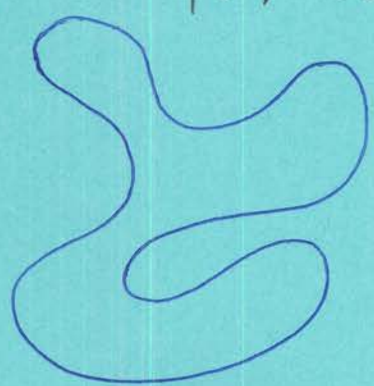
Can we go the other way? That is, if $\vec{F} = \langle P, Q \rangle$, does $P_y = Q_x$ mean $\vec{F}$ is conservative?

The answer is yes, but only in certain cases.
First, we need two things:

<u>Def</u>: A <u>simple curve</u> is one that does not cross itself anywhere between its endpoints; i.e., if $\vec{r}(t)$, $a \le t \le b$, is a parametrization of the curve, then $\vec{r}(t_1) \ne \vec{r}(t_2)$ for any $a < t_1 < t_2 < b$.

<u>Ex</u>:

simple, closed



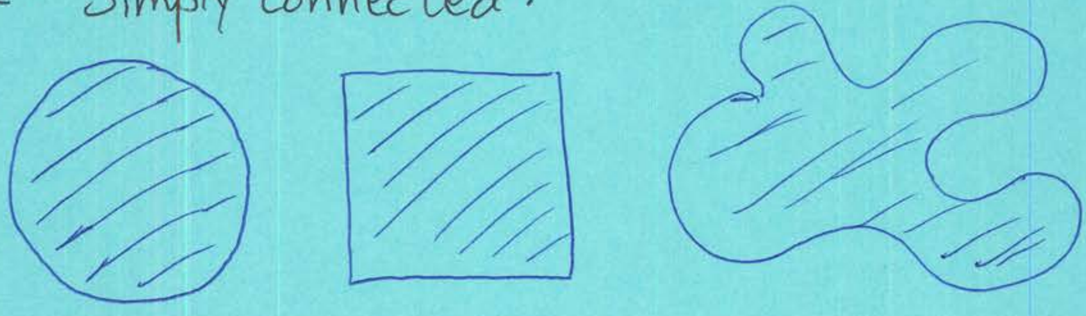
not simple, closed

simple, not closed

not simple, not closed

<u>Def</u>: A region D is called <u>simply connected</u> if D is connected and every simple closed curve in D encloses only points in D.

Ex: Simply connected:



Not simply connected:

Finally:

Thm: If $\vec{F} = \langle P, Q \rangle$ is a vector field on an open, simply connected region D, then if P and Q are C^1 and $P_y = Q_x$, we have that $\vec{F}$ is conservative.

Now, how do we go about finding the potential function? We know, if a vector field is conservative, then $\vec{F} = \nabla f = \langle f_x, f_y \rangle$. So, if f is a potential for $\vec{F} = \langle P, Q \rangle$, then we should have $P = f_x$ & $Q = f_y$. So $f = \int P\,dx$ and $f = \int Q\,dy$. Now doing either one of these integrals by itself isn't enough.

This is because, for example, we get an arbitrary function of y, $h(y)$, when we integrate w.r.t. x: $f(x,y)=\int P\,dx = p(x,y)+h(y)$. We fix this, since $f_y = Q$, by taking the derivative w.r.t. y and setting it equal to Q to find $h(y)$.

Ex: Determine whether the vector field

$$\vec{F}(x,y)=\langle e^x \sin y + 2xy^3,\, e^x \cos y + 3x^2y^2\rangle$$

is conservative. If so, find a potential function.

Sol: It's easiest to just try to find a potential.

Here, $P=e^x\sin y + 2xy^3$ & $Q = e^x \cos y + 3x^2y^2$.

Let's integrate Q:

$$f(x,y)=\int Q\,dy = \int (e^x\cos y + 3x^2y^2)\,dy = e^x \sin y + x^2y^3 + g(x).$$

To find g, take the derivative of f w.r.t. x and set it equal to P:

$$f_x(x,y)= e^x \sin y + 2xy^3 + g'(x) = P = e^x\sin y + 2xy^3 \Rightarrow g'(x)=0$$

$\Rightarrow g(x)$ is a constant, so we can choose any one.

So, a potential function for $\vec{F}$ is

$$f(x,y) = e^x \sin y + x^2 y^3,$$

thus $\vec{F}$ is conservative.

(The domain of $\vec{F}$ is simply connected, so there isn't a problem.)

$\diamond$

We can also do this same thing in 3-variables. We also need simply connected regions here, but that's harder to define in 3-D. Yet, if a vector field on $\mathbb{R}^3$ is conservative, the process for finding the potential is similar, e.g., if $\vec{F} = \langle P, Q, R \rangle$ is conservative, then $\vec{F} = \langle f_x, f_y, f_z \rangle$, and so $P = f_x$, $Q = f_y$, $R = f_z$.

If we start by integrating w.r.t. x, then $f(x,y,z) = \int P\,dx = p(x,y,z) + g(y,z)$. Now, we compare with Q: $f_y = \frac{\partial p}{\partial y} + \frac{\partial g}{\partial y} = Q$. Then we can solve for $\frac{\partial g}{\partial y}$ and integrate w.r.t. y to get g:

$$g(y,z) = \int \frac{\partial g}{\partial y}(y,z)\,dy = G(y,z) + h(z) \quad \left(G \text{ is a known function}\right)$$

Then, we differentiate w.r.t. z and solve for h:

$$f(x,y,z) = p(x,y,z) + g(y,z) = p(x,y,z) + G(y,z) + h(z)$$

$$\Rightarrow f_z = \frac{\partial p}{\partial z} + \frac{\partial G}{\partial z} + h' = R$$

From here we can solve for h and finally get f.

<u>Ex</u>: Determine whether the vector field

$$\vec{F}(x,y,z) = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$$

is conservative. If it is, find a potential.

<u>Sol</u>: As before, the best way to check if $\vec{F}$ is conservative is to try to find a potential.

$$f = \int P\,dx = \int e^{yz}\,dx = xe^{yz} + g(y,z)$$

$$f_y = xze^{yz} + \frac{\partial g}{\partial y} = Q = xze^{yz} \Rightarrow \frac{\partial g}{\partial y}(y,z) = 0$$

$$\Rightarrow g(y,z) = \int 0\,dy = h(z) \Rightarrow f = xe^{yz} + h(z)$$

$$f_z = xye^{yz} + h'(z) = R = xye^{yz} \Rightarrow h'(z) = 0 \Rightarrow h(z) = c$$

So, a potential is $f(x,y,z) = xye^{yz}$

$\diamond$

Define the <u>curl</u> of a vector field as

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

We do have the following theorems in 3D:

<u>Fact</u>: If $\vec{F}$ is conservative, then curl $\vec{F} = \vec{0}$

<u>Thm</u>: If $\vec{F} = \langle P, Q, R \rangle$ is a vector field on an open, simply connected region in $\mathbb{R}^3$, and if P, Q, and R are C^1, then if curl $\vec{F} = \vec{0}$, we have that $\vec{F}$ is a conservative vector field